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AN ALGORITHM FOR THE ROTA STRAIGHTENING FORMULA

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The algorithm here constructed for the Rota straightening formula is a combination of sorting techniques and of algebraic methods closely related to the Rutherford–Young Natural representation of the Symmetric group.

1. Introduction

Characterization of invariant forms led Rota to introduce the notions of bitableau and bideterminant, and to construct a remarkable basis of the algebra of invariant forms. The elements of that basis are the so-called standard bideterminants. Each of them is associated with a standard bitableau. The *straightening formula* essentially says that every form is a linear combination with integral coefficients of standard bideterminants. The coefficients of this formula—below referred as straightening coefficients—can be calculated by means of the Capelli operators. However, no effective algorithm has ever been proposed to achieve the calculations. It is the purpose of this paper to make up one.

The algorithm is a combination of an algebraic method developed by Rutherford in the case of single tableaux, and also of a classical sorting technique dear to computer scientists (see Lemma 1 below).

Tables of straightening coefficients up to order 5 have been published in the appendix of [2].

2. Standardization

Let $(\lambda)=(\lambda_1,\ldots,\lambda_p)$ be a *partition* of the integer n; that is, (λ) is a finite sequence of positive integers with

$$\lambda_1+\cdots+\lambda_p=n \quad\text{and}\quad \lambda_1\geqslant\cdots\geqslant\lambda_p>0.$$

The *shape* of (λ) is denoted by (λ); it is the set of integral points (i, j) in the plane with $1\leqslant j\leqslant p$ and $1\leqslant i\leqslant \lambda_j$.

A *Young tableau* (or simply a tableau) of shape (λ) with values in the set E is a mapping of the shape (λ) into E. For example, T_1, T_2, T_3 and T_4 are Young tableaux of shape $(4,3,3,2,1)$ with integral values:

$$T_1=\begin{matrix}4\\3\;4\\1\;2\;5\\2\;1\;3\\3\;2\;3\;4\end{matrix},\quad T_2=\begin{matrix}4\\3\;4\\1\;3\;2\\2\;1\;3\\3\;2\;5\;4\end{matrix},\quad T_3=\begin{matrix}4\\3\;4\\1\;2\;3\\1\;2\;3\\2\;3\;4\;5\end{matrix},\quad T_4=\begin{matrix}4\\3\;4\\2\;3\;4\\1\;2\;3\\1\;2\;3\;5\end{matrix}$$

A Young tableau is *row-injective* if no entry is repeated in any of its rows. If E is a totally ordered set (which is always the case in this paper), a Young tableau is *normal* if the entries in each row are increasing from left to right; this tableau is *standard* if it is normal and if the entries in each column are nondecreasing from bottom to top. In the previous example, T_1 is not row-injective, while T_2, T_3 and T_4 are row-injective; moreover T_3 is normal and T_4 is standard.

If T is a tableau, let us denote by ${}^{C}T$ the tableau obtained from T by writing the entries in each column in nondecreasing order. If T is row-injective, rewriting the entries in each row in increasing order yields a normal tableau, called the *normalized* of T and denoted by ${}^{R}T$.

Lemma 1. *Let T be a row-injective tableau. Then the tableau ${}^{C}({}^{R}T)$ is standard.*

It is equivalent to prove that when T is normal so is the tableau ${}^{C}T$.

First, assume that T is a rectangular tableau with p rows and q columns. We shall construct by induction a sequence of rectangular tableaux

$$T^{(0)}=T,\quad T^{(1)},\ldots,T^{(q)},$$

which differ from T only by the order of their entries in each column. Moreover, these tableaux satisfy the following two properties:

(i) for $1\leqslant k\leqslant q$, the entries of the first k columns of $T^{(k)}$ are nondecreasing upwards;

(ii) for $0\leqslant k\leqslant q$, the tableau $T^{(k)}$ is normal.

If such a sequence is constructed, the tableau $T^{(q)}$ is normal by (ii); as $T^{(q)}$ and ${}^{C}T$ are identical, the proof of the lemma is achieved when T is rectangular.

We define by $t_{i,j}^{(k)}$ the entry in row i and column j of $T^{(k)}$.

The first step of the induction is the construction of $T^{(1)}$: choose a permutation σ that satisfies

$$t_{\sigma 1,1}\leqslant t_{\sigma 2,1}\leqslant\cdots\leqslant t_{\sigma p,1}.$$

(Note that in general such a permutation is not unique.) Then define

$$t_{i,j}^{(1)}=t_{\sigma i,j}\quad\text{for } 1\leqslant i\leqslant p \text{ and } 1\leqslant j\leqslant q.$$

Except for the order, the rows of $T^{(1)}$ are identical to the rows of T, so that $T^{(1)}$ is normal. Consequently, it satisfies properties (i) and (ii).

Now suppose $T^{(k)}$ has been constructed. In the next step a classical and simple sorting algorithm will be used: let $a=(a_1, a_2, \ldots, a_p)$ be a given sequence of elements of a totally ordered set. Choose, if possible, two entries a_i and a_{i+1} such that $a_i > a_{i+1}$. Then transpose these entries and call the new sequence

$$a' = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_p).$$

Repeat this with a' instead of a. After a finite number of such operations, the resulting sequence is in increasing order.

If column $k+1$ of $T^{(k)}$ is already in nondecreasing order, define $T^{(k+1)} = T^{(k)}$.

If this column is not in nondecreasing order, there exists an index u such that $t^{(k)}_{u,k+1} > t^{(k)}_{u+1,k+1}$. We define a tableau T' by its entries:

$$t'_{i,j} = t^{(k)}_{i,j} \quad \text{if } i \neq u,\ i \neq u+1 \text{ and } 1 \leq j \leq q;$$

$$\begin{cases} t'_{u,j} = t^{(k)}_{u,j} & \text{if } 1 \leq j \leq k, \\ t'_{u,j} = t^{(k)}_{u+1,j} & \text{if } k+1 \leq j \leq q; \end{cases}$$

$$\begin{cases} t'_{u+1,j} = t^{(k)}_{u+1,j} & \text{if } 1 \leq j \leq k, \\ t'_{u+1,j} = t^{(k)}_{u,j} & \text{if } k+1 \leq j \leq q. \end{cases}$$

The tableau T' differs from $T^{(k)}$ only in rows u and $u+1$:

$$T' = \begin{matrix} \vdots & & \vdots & \vdots & & \vdots \\ t^{(k)}_{u+1,1} & \cdots & t^{(k)}_{u+1,k} & t^{(k)}_{u,k+1} & \cdots & t^{(k)}_{u,q} \\ t^{(k)}_{u,1} & \cdots & t^{(k)}_{u,k} & t^{(k)}_{u+1,k+1} & \cdots & t^{(k)}_{u+1,q} \\ \vdots & & \vdots & \vdots & & \vdots \end{matrix}$$

As $T^{(k)}$ is normal, and by hypothesis $t^{(k)}_{u+1,k+1} < t^{(k)}_{u,k+1}$, the entries in row $u+1$ of T' are increasing. On the other hand, the following inequalities hold:

$$t^{(k)}_{u,k} \leq t^{(k)}_{u+1,k} \qquad \text{(since } T^{(k)} \text{ satisfies (i))}$$

and

$$t^{(k)}_{u+1,k} < t^{(k)}_{u+1,k+1} \qquad \text{(since } T^{(k)} \text{ is normal).}$$

Thus we have proved that the entries are increasing in row u of T'. Then T' satisfies properties (i) and (ii). In the construction of $T^{(k+1)}$ we now use T' instead of $T^{(k)}$. After a finite number of such steps, by using the above mentioned sorting argument, column $k+1$ of the resulting tableau is properly ordered. This last tableau is called $T^{(k+1)}$.

The lemma is proved provided T is rectangular.

Now suppose that T is a normal tableau of shape $(\lambda) = (\lambda_1, \ldots, \lambda_p)$. Let us assume that the entries of T are natural numbers, and let s be the greatest number in T. We define the rectangular tableau U by

$$u_{i,j}=t_{i,j} \quad \text{if } 1\leq i\leq p \text{ and } 1\leq j\leq \lambda_i;$$

$$u_{i,j}=s+j \quad \text{if } 1\leq i\leq p \text{ and } \lambda_i+1\leq j\leq \lambda_1:$$

Clearly U is normal and the entries greater than s are in the same places in U and ${}^{C}U$. In other words, to reorder the columns of U, we just need to reorder the columns of T, so that ${}^{C}T$ is a part of the normal tableau ${}^{C}U$. Consequently ${}^{C}T$ is normal. This concludes the proof of the lemma.

The following definition is then justified: when T is row-injective, the tableau ${}^{C}({}^{R}T)$ is called the *standardized* of T; it is denoted by ${}^{S}T$.

3. The coefficients ξ

We begin this section by a notation. Let T be a row-injective tableau. The row i of the normalized ${}^{R}T$ of T is the image of the corresponding row of T under the action of a permutation σ_i. The product $\prod_i (-1)^{\sigma_i}$ of the signatures of all these permutations is denoted by $\nu(T)$.

Let T_1 and T_2 be two tableaux of the same shape (λ) and same content (that is, the number of occurrences of any given element is the same in T_1 and T_2). Suppose that T_1 is *normal* and T_2 is *standard*. If at least one row-injective tableau T exists with the property that ${}^{R}T=T_1$ *and* ${}^{C}T=T_2$, then define the *coefficient* ξ of T_1 and T_2 by

$$\xi(T_1,T_2)=\sum\{\nu(T):\ {}^{R}T=T_1,\ {}^{C}T=T_2\}.$$

If no such a tableau exists, let

$$\xi(T_1,T_2)=0.$$

Example. Suppose

$$T_1=\begin{matrix}2&3&\\2&3&4\\1&3&4\end{matrix} \quad \text{and} \quad T_2=\begin{matrix}3&4&\\2&3&4\\1&2&3\end{matrix}.$$

Three tableaux T can be found such that ${}^{R}T=T_1$ and ${}^{C}T=T_2$:

$$T=\begin{matrix}3&2&\\2&4&3\\1&3&4\end{matrix}; \quad T'=\begin{matrix}3&2&\\2&3&4\\1&4&3\end{matrix}; \quad T''=\begin{matrix}2&3&\\3&2&4\\1&4&3\end{matrix}.$$

Each of these tableaux differs from T_1 by two transpositions. Hence we have $\nu(T)=\nu(T')=\nu(T'')=1$, and finally $\xi(T_1, T_2)=3$. If we now take

$$U = \begin{matrix} 1 & 3 & \\ 2 & 3 & 4, \\ 2 & 3 & 4 \end{matrix}$$

we find two tableaux such that ${}^{R}T=U$ and ${}^{C}T=T_2$, namely

$$\begin{matrix} 1 & 3 & \\ 3 & 2 & 4 \\ 2 & 4 & 3 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 3 & \\ 2 & 4 & 3. \\ 3 & 2 & 4 \end{matrix}$$

Thus $\xi(U, T_2)=2$.

Let us assign a total order on the tableaux of same shape and content. Associate with the tableau T the sequence $(t_{11}, t_{21}, \ldots, t_{12}, t_{22}, \ldots)$ obtained by reading off its entries successively in each column upwards. The tableaux are then ordered according to the lexicographical order of their associated column sequences.

Example. For the tableaux of the preceding example the associated column sequences are respectively:

$$T_1 \to (1,2,2,3,3,3,4,4),$$
$$T_2 \to (1,2,3,2,3,4,3,4),$$
$$U \to (2,2,1,3,3,3,4,4).$$

Hence $T_1 \leqslant T_2 \leqslant U$.

Theorem 1. *Let T_1 be a normal tableau and T_2 be a standard tableau of the same shape and content. Then*

(i) $\xi(T_1, T_2) \neq 0$ *implies* ${}^{S}T_1 \leqslant T_2$;

(ii) $\xi(T_1, {}^{S}T_1)=1$.

Proof. (i) Let T be such that ${}^{R}T=T_1$ and ${}^{C}T=T_2$. Let v be the first column where T_1 and T differ. In column v of T_1 let x be the smallest entry which is not in the same place in T. As ${}^{R}T=T_1$, this x is replaced in T by an index y from the same row, on the right of x. The tableau T_1, is normal so that $x<y$. The ordered columns of T and T_1 are identical up to the vth. In column v the ordered indices are the same up to a certain index, which is x in ${}^{C}T_1$ and y in ${}^{C}T$. Therefore

$${}^{S}T_1 = {}^{C}T_1 \leqslant {}^{C}T = T_2.$$

(ii) In fact, if T differs from T_1, the existence of such an x proves that ${}^{S}T_1$ is strictly smaller than T_2. When $T_2={}^{S}T_1$, there is evidently only one tableau T satisfying the required conditions. This unique tableau is T_1. It is normal, so that $\nu(T_1)=1$. This concludes the proof of Theorem 1.

4. The straightening formula

We now apply the preceding results to Rota's bideterminants. For every question related to this theory, and particularly for the proofs of Theorems 2 and 3, we refer to [1, 2]. We just mention below fundamental definitions and properties.

Let $\mathbb{K}$ be a field and let $\mathscr{X}=\{x_1,\ldots,x_n\}$ and $\mathscr{U}=\{u_1,\ldots,u_p\}$ be two alphabets; a *form* is an element of the algebra P of polynomials over $\mathbb{K}$ in the np indeterminates $(x_i \mid u_j)$. Considering two finite sequences $(x_{i_1},\ldots,x_{i_d})$ and $(u_{j_1},\ldots,u_{j_d})$ of letters from $\mathscr{X}$ and $\mathscr{U}$ respectively, their *inner product*

$$(x_{i_1}\cdots x_{i_d} \mid u_{j_1}\cdots u_{j_d})$$

is the polynomial in P defined by:

$$(x_{i_1}\cdots x_{i_d} \mid u_{j_1}\cdots u_{j_d})=\sum_{\sigma\in\mathfrak{S}_d}(-1)^{\sigma}(x_{i_{\sigma 1}} \mid u_{j_1})\cdots(x_{i_{\sigma d}} \mid u_{j_d}).$$

It is an antisymmetric form.

A *bitableau* is a pair $[T, T']$ of Young tableaux of the same shape, where T (resp. T') has values in $\mathscr{X}$ (resp. $\mathscr{U}$). This bitableau is *normal* (resp. *standard*) if both T and T' are normal (resp. standard). The *bideterminant* $(T \mid T')$ is the product of the inner products of each row of T with the corresponding row in T'; it is an element of P. By reordering the entries in each row of T and T', it is clear that $(T \mid T')$ is either zero or, up to a change of sign, the bideterminant of a normal bitableau.

Theorem 2 (The straightening formula). *The bideterminant of a bitableau is a linear combination, with integral coefficients, of bideterminants of standard bitableaux of the same or longer shape.*

In fact the straightening coefficients are *unique.* The standard bideterminants form a basis of P. To prove this last result, Rota introduced the Capelli operators. Let us consider another algebra of forms, the indeterminates of which are $(s_i \mid t_j)$, where $\mathscr{S}=\{s_1,s_2,\ldots\}$ and $\mathscr{T}=\{t_1,t_2,\ldots\}$ are two new alphabets.

Suppose $[T, T']$ is a standard bitableau of shape (λ). Let $\tilde{\lambda}_l$ be the height of the lth column of T and let $\alpha_i(l)$ (resp. $\beta_j(l)$) be the number of occurrences of x_i (resp. u_j) in the lth column of T (resp. T').

Let $[U, U']$ be another bitableau with the same content. We denote by $C(T, T')(U \mid U')$ the form in the indeterminates $(s_i \mid t_j)$ which is the sum of the bideterminants obtained by substituting in all possible ways letters s_l to $\alpha_i(l)$ letters x_i and letters t_l to $\beta_j(l)$ letters u_j, $1 \leq l \leq \lambda_1$, $1 \leq i \leq n$, $1 \leq j \leq p$. The operator $C(T, T')$ is called the *Capelli operator* of $[T, T']$.

Theorem 3. *Let $[T, T']$ and $[U, U']$ be two standard bitableaux of shape (λ) and (μ) with the same content. Then*

(i) $C(T, T')(T \mid T') \neq 0$;

(ii) *if (μ) is longer than (λ), then* $C(T, T')(U \mid U') = 0$;

(iii) *if $(\lambda) = (\mu)$, and if $C(T, T')(U \mid U') \neq 0$, then $[U, U']$*

is smaller than $[T, T']$ in the lexicographical order of their associated column sequences.

The action of Capelli operators on bideterminants of the same shape can be expressed by means of the coefficients ξ. The results of the preceding sections are applied provided the variables are identified with their subscripts.

Theorem 4. *Let $[T, T']$ be a standard bitableau of shape (λ), and let $[U, U']$ be a normal bitableau with the same shape and content. Then*

$$C(T, T')(U \mid U') = \xi(U, T)\xi(U', T')C(T, T')(T \mid T').$$

Proof. Let $(V \mid V')$ be one of the bideterminants of shape (λ) which appears as a term in the sum $C(T, T')(U \mid U')$. This bideterminant can be zero, in case a letter is repeated inside a row. If it is not zero, a mere application of the pigeon-hole principle shows that the letters s_l are to be found in the first $\tilde{\lambda}_l$ rows of V. In other words, if we call $(\theta \mid \theta')$ the bideterminant

$$C(T, T')(T \mid T') = \left(\begin{array}{cccc|cccc} s_1 & & & & t_1 & & & \\ \cdot & s_2 & . & & \cdot & t_2 & . & \\ & & \cdot & \cdot\ s_{\lambda_1} & \cdot & & \cdot & \cdot\ t_{\lambda_1} \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ s_1 & s_2 & & s_{\lambda_1} & t_1 & t_2 & & t_{\lambda_1} \\ s_1 & s_2 & \ldots & s_{\lambda_1} & t_1 & t_2 & \ldots & t_{\lambda_1} \end{array}\right),$$

we have the relations ${}^{\mathrm{R}}V = \theta$ and ${}^{\mathrm{R}}V' = \theta'$. As the inner product is an antisymmetric form, we obtain the identity

$$(V \mid V') = \nu(V)\nu(V')(\theta \mid \theta').$$

The letters of $(U \mid U')$ replaced by s_l are situated in the first $\tilde{\lambda}_l$ rows of U; they are exactly the entries of the lth column of T. The permutation mapping $[\theta, \theta']$ to $[V, V']$, sends a certain bitableau $[W, W']$ to $[U, U']$. It is clear that the bitableau $[W, W']$ verifies $[{}^RW, {}^RW'] = [U, U']$ and $[{}^CW, {}^CW'] = [T, T']$.

Conversely, from such a bitableau $[W, W']$ it is possible to find again $[V, V']$. As $\nu(W) = \nu(V)$ and $\nu(W') = \nu(V')$, we obtain

$$C(T, T')(U \mid U') = \sum (V \mid V') = \sum \nu(W)\nu(W')(\theta \mid \theta'),$$

where the summation is over all bitableaux $[W, W']$ such that $[{}^RW, {}^RW'] = [U, U']$ and $[{}^CW, {}^CW'] = [T, T']$. This last identity is equivalent to the required formula.

The preceding theorem allows us to calculate the straightening coefficients relative to the same shape by means of coefficients ξ.

Suppose the l standard bitableaux of shape (λ) are ordered according to the lexicographical order of their associated column sequences:

$$[T_1, T_1'] \le [T_2, T_2'] \le \cdots \le [T_l, T_l'].$$

Let us write for convenience $C_i = C(T_i, T_i')$, $\xi_i = \xi(U, T_i)\xi(U', T_i')$ and $\xi_{ij} = \xi(T_i, T_j)\xi(T_i', T_j')$. From Theorem 1, the $l \times l$ matrix

$$\Xi = (\xi_{ij})$$

is triangular with all diagonal entries equal to 1.

By applying the straightening formula, we have

$$(U \mid U') = \sum_{1 \le i \le l} \alpha_i (T_i \mid T_i') + A,$$

where A is a linear combination of bideterminants of strictly longer shapes than (λ). Remark that all bideterminants $C_i(T_i \mid T_i')$ are equal to the same standard bideterminant $(\theta \mid \theta')$. The operator C_j acts as follows

$$C_j(U \mid U') = \sum_{1 \le i \le l} \alpha_i C_j(T_i \mid T_i')$$

since C_jA vanishes by Theorem 3(ii). By Theorem 4 we can write

$$\xi_j(\theta \mid \theta') = \sum_{1 \le i \le l} \alpha_i \xi_{ij}(\theta \mid \theta').$$

So we obtain the formula giving the straightening coefficients α_i:

$$(\alpha_1\alpha_2\cdots\alpha_l)=(\xi_1\xi_2\cdots\xi_l)\Xi^{-1}.$$

But the matrix Ξ^{-1} is triangular with 1's on the diagonal and from Theorem 1 we have $\xi_i=0$ if $[T_i, T_i']<[{}^CU, {}^CU']$ and $\xi_i=1$ if $[T_i, T_i']=[{}^CU, {}^CU']$. This yields the following theorem.

Theorem 5. *Let $[U, U']$ be a normal bitableau. Then $({}^CU \mid {}^CU')$ is the smallest standard bideterminant which appears in the decomposition of $(U \mid U')$. Moreover, its coefficient is* 1.

Example. The bitableaux

$$[T_1, T_1']=\begin{bmatrix}2\ 3 & 2\ 3\\ 2\ 3\ 4, & 2\ 3\ 4\\ 1\ 3\ 4 & 1\ 3\ 4\end{bmatrix};\quad [T_2, T_2']=\begin{bmatrix}2\ 3 & 3\ 4\\ 2\ 3\ 4, & 2\ 3\ 4\\ 1\ 3\ 4 & 1\ 2\ 3\end{bmatrix};$$

$$[T_3, T_3']=\begin{bmatrix}3\ 4 & 2\ 3\\ 2\ 3\ 4, & 2\ 3\ 4\\ 1\ 2\ 3 & 1\ 3\ 4\end{bmatrix};\quad [T_4, T_4']=\begin{bmatrix}3\ 4 & 3\ 4\\ 2\ 3\ 4, & 2\ 3\ 4\\ 1\ 2\ 3 & 1\ 2\ 3\end{bmatrix}$$

are the only standard bitableaux with this shape and content. It is clear that, in order to determine the corresponding matrix Ξ, we just have to calculate the coefficient $\xi(T_1, T_3)$. In the first example of section 3 we have already determined this coefficient:

$$\xi(T_1, T_3)=3.$$

So we obtain

$$\Xi=\begin{pmatrix}1 & 3 & 3 & 9\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 1\end{pmatrix}\quad \text{and}\quad \Xi^{-1}=\begin{pmatrix}1 & -3 & -3 & 9\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Let

$$[U, U']=\begin{bmatrix}1\ 3 & 1\ 3\\ 2\ 3\ 4, & 2\ 3\ 4\\ 2\ 3\ 4 & 2\ 3\ 4\end{bmatrix}.$$

As ${}^CU=T_1$ *and* ${}^CU'=T_1'$, we have $\xi_1=1$. The coefficients $\xi(U, T_3)=\xi(U', T_2')$

have been calculated in the above-mentioned example and found to be 2. Hence

$$(\xi_1\ \xi_2\ \xi_3\ \xi_4) = (1\ 2\ 2\ 4)$$

and

$$(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4) = (1\ 2\ 2\ 4)\begin{pmatrix} 1 & -3 & -3 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1\ -1\ -1\ 1).$$

Finally

$$(U \mid U') = (T_1 \mid T_1') - (T_2 \mid T_2') - (T_3 \mid T_3') + (T_4 \mid T_4') + A.$$

5. The substitution operators

Let $[U, U']$ be a bitableau; let $S(s_j, x_i)[U, U']$ be the bitableau obtained by replacing in U the letter x_i by s_j whenever it appears. The bideterminant of this last bitableau can be denoted by $S(s_j, x_i)(U \mid U')$, and the *substitution operator* $S(s_j, x_i)$ is, of course, a linear operator (we naturally suppose that we have extended the algebra P by adjoining such indeterminates as $(s_j \mid u_k)$). Particularly, if $(U \mid U') = \sum_k \alpha_k (T_k \mid T_k')$ is the decomposition of $(U U')$, then

$$S(s_j, x_i)(U \mid U') = \sum_k \alpha_k S(s_j, x_i)(T_k \mid T_k').$$

By means of these operators, the decomposition of any bideterminant can be deduced from the decomposition of an injective bideterminant. Consider a bitableau $[U, U']$ with content

$$((\alpha_1, \alpha_2, \ldots, \alpha_n), (\beta_1, \beta_2, \ldots, \beta_p)),$$

i.e. the letter x_i (resp. u_j) occurs α_i (resp. β_j) times in U (resp. U'). Now construct an injective bitableau $[V, V']$ as follows:

(1) If $1 \leq i \leq n$ the x_i's in U are replaced by α_i distinct letters $s_{\alpha_1+\alpha_2+\cdots+\alpha_{i-1}+k}$, $1 \leq k \leq \alpha_i$; furthermore the index of the row where the variable $s_{\alpha_1+\cdots+\alpha_{i-1}+k}$, $1 \leq k \leq \alpha_i$, occurs increases with k.

(2) The same is done for U', by replacing the u_*'s by t_*'s.

As before, in the examples, only the subscripts of the variables are written.

Example. Let us take

$$[U, U'] = \begin{bmatrix} 1\ 3 & & 1\ 3 & \\ 2\ 3\ 4, & & 2\ 3\ 4 & \\ 2\ 3\ 4 & & 2\ 3\ 4 & \end{bmatrix}.$$

The corresponding injective tableau is

$$[V, V'] = \begin{bmatrix} 1\ 6 & & 1\ 6 & \\ 3\ 5\ 8, & & 3\ 5\ 8 & \\ 2\ 4\ 7 & & 2\ 4\ 7 & \end{bmatrix}.$$

If we know the injective bitableau $[V, V']$ and the content (α, β) of $[U, U']$, we can reconstruct this last bitableau: by $S_{(\alpha,\beta)}$ we denote the commutative product

$$\prod_{1\le i\le n}\ \prod_{1\le j\le \alpha_i} S(x_i, s_{\alpha_1+\cdots+\alpha_{i-1}+j}) \prod_{1\le l\le p}\ \prod_{1\le m\le \beta_l} S(u_l, t_{\beta_1+\cdots+\beta_{l-1}+m}).$$

Then $[U, U'] = S_{(\alpha,\beta)}[V, V']$. The proof of the following lemma is quite clear.

Lemma 2. *If $[T, T']$ is a standard injective bitableau, then the bitableau $S_{(\alpha,\beta)}[T, T']$ is either standard or its bideterminant is zero.*

The action of substitution operators on straightening identities involving injective bideterminants leads to similar identities for non injective bideterminants.

Example. Let

$$(U \mid U') = \left(\begin{matrix} 1\ 3 \\ 2\ 3\ 4 \end{matrix} \,\middle|\, \begin{matrix} 3\ 4 \\ 1\ 2\ 3 \end{matrix}\right).$$

The content is then

$$(\alpha, \beta) = ((1, 1, 2, 1), (1, 1, 2, 1))$$

and

$$(U \mid U') = S_{(\alpha,\beta)}\left(\begin{matrix} 1\ 4 \\ 2\ 3\ 5 \end{matrix} \,\middle|\, \begin{matrix} 4\ 5 \\ 1\ 2\ 3 \end{matrix}\right).$$

Suppose we know the decomposition

$$\left(\begin{matrix} 1\ 4 \\ 2\ 3\ 5 \end{matrix} \,\middle|\, \begin{matrix} 4\ 5 \\ 1\ 2\ 3 \end{matrix}\right) = \left(\begin{matrix} 2\ 4 \\ 1\ 3\ 5 \end{matrix} \,\middle|\, \begin{matrix} 4\ 5 \\ 1\ 2\ 3 \end{matrix}\right) - \left(\begin{matrix} 3\ 4 \\ 1\ 2\ 5 \end{matrix} \,\middle|\, \begin{matrix} 4\ 5 \\ 1\ 2\ 3 \end{matrix}\right) - \left(\begin{matrix} 4\ 5 \\ 1\ 2\ 3 \end{matrix} \,\middle|\, \begin{matrix} 4\ 5 \\ 1\ 2\ 3 \end{matrix}\right)$$

$$+ \left(\begin{matrix} 4 \\ 1\ 2\ 3\ 5 \end{matrix} \,\middle|\, \begin{matrix} 4 \\ 1\ 2\ 3\ 5 \end{matrix}\right) - \left(\begin{matrix} 4 \\ 1\ 2\ 3\ 5 \end{matrix} \,\middle|\, \begin{matrix} 5 \\ 1\ 2\ 3\ 4 \end{matrix}\right).$$

By applying $S_{(\alpha,\beta)}$ we find

$$(U\,|\,U')=\begin{pmatrix}2&3&&\\1&3&4\end{pmatrix}\!\!\begin{vmatrix}3&4&\\1&2&3\end{vmatrix}-\begin{pmatrix}3&3&\\1&2&4\end{pmatrix}\!\!\begin{vmatrix}3&4&\\1&2&3\end{vmatrix}-\begin{pmatrix}3&4&\\1&2&3\end{pmatrix}\!\!\begin{vmatrix}3&4&\\1&2&3\end{vmatrix}$$

$$+\begin{pmatrix}3&&&\\1&2&3&4\end{pmatrix}\!\!\begin{vmatrix}3&&&\\1&2&3&4\end{vmatrix}-\begin{pmatrix}3&&&\\1&2&3&4\end{pmatrix}\!\!\begin{vmatrix}4&&&\\1&2&3&3\end{vmatrix};$$

the second and the fifth bideterminant in the right side of this last identity vanish, so that we obtain the straightening decomposition

$$(U\,|\,U')=\begin{pmatrix}2&3&\\1&3&4\end{pmatrix}\!\!\begin{vmatrix}3&4&\\1&2&3\end{vmatrix}-\begin{pmatrix}3&4&\\1&2&3\end{pmatrix}\!\!\begin{vmatrix}3&4&\\1&2&3\end{vmatrix}+\begin{pmatrix}3&&&\\1&2&3&4\end{pmatrix}\!\!\begin{vmatrix}3&&&\\1&2&3&4\end{vmatrix}.$$

6. The action of permutations

Let (λ) be a partition of n and $[V, V']$ be an injective bitableau of shape (λ). A permutation $\sigma\in\mathfrak{S}_n$ can act in two possible ways: on V or on V'. The tableau V is obtained by replacing in V the index i by σi, $1\leqslant i\leqslant n$. This action can obviously be linearly transposed to bideterminants.

To solve the problem of the decomposition of an injective bideterminant $(V\,|\,V')$ we can suppose without loss of generality that V' is standard: to decompose $(V\,|\,V')$ determine σ such that $\sigma V'$ is standard. Then

$$(V\,|\,\sigma V')=\sum_i \alpha_i(T_i\,|\,T_i').$$

Hence

$$(V\,|\,V')=\sum_i \alpha_i(T_i\,|\,\sigma^{-1}T_i').$$

The role played by both tableaux in a bideterminant is symmetric. Therefore, the decomposition of $(T_i\,|\,\sigma^{-1}T_i')$ can be deduced from that of $(\sigma^{-1}T_i'\,|\,T_i)$ (which has the required form) by permuting the tableaux.

Let $T'_{\max}$ be the greatest standard tableau of shape (λ). We can then restrict ourselves to the case where $V'=T'_{\max}$ by choosing σ appropriately. Let $[V, T'_{\max}]$ be a bitableau of shape (λ). From Theorems 1 and 5, the only bideterminants of same shape appearing in the decomposition of $(V\,|\,T'_{\max})$ are the bideterminants $(T_i\,|\,T'_{\max})$ where T_i is greater than ${}^{S}V$. The coefficients ξ arising are those relative to tableaux V and T_i. The corresponding matrix Ξ is exactly that which arises in the classical theory of the normal representation of the symmetric group, as developed by Rutherford.

Let then $[V, V']$ be an injective bitableau of shape (λ). By f_λ denote the number of standard injective tableaux of shape (λ). If

$$(V \mid T'_{\max}) = \sum_{1 \le i \le f_\lambda} a_i (T_i \mid T'_{\max}) + A$$

and

$$(T_{\max} \mid V') = \sum_{1 \le i \le f_\lambda} b_i (T_{\max} \mid T'_i) + B,$$

where A and B are linear combinations of strictly longer bideterminants, then, by applying a permutation to $T'_{\max}$, we have

$$(V \mid T') = \sum_{1 \le i \le f_\lambda} a_i (T_i \mid T') + A'$$

for any injective tableau T'. In particular we have

$$(V \mid V') = \sum_{1 \le i \le f_\lambda} a_i (T_i \mid V') + A'$$

and

$$(T_i \mid V') = \sum_{1 \le j \le f_\lambda} b_j (T_i \mid T'_j) + B'.$$

Finally

$$(V \mid V') = \sum_{1 \le i,\, j \le f_\lambda} a_i b_j (T_i \mid T'_j) + C,$$

where C is a linear combination of strictly longer bideterminants.

7. The longer shapes

Suppose we have ordered the $\sum_{(\lambda)} f_\lambda^2 = n!$ injective standard bitableaux according to their length and lexicographically inside each shape. Suppose we know the straightening coefficients of $(V \mid V')$, a_1, a_2, up to a_{p-1}:

$$(V \mid V') = \sum_{1 \le i \le p-1} a_i (T_i \mid T'_i) + A.$$

To determine a_p, use the Capelli operator $C_p = C(T_p, T'_p)$. As a consequence of

Theorem 3 we have

$$a_pC_p(T_p \mid T'_p) = C_p(V \mid V') - \sum_{1\leq i\leq p-1} a_iC_p(T_i \mid T'_i).$$

Let (μ) be the shape of $(T_p \mid T'_p)$. We know that

$$C_p(T_p \mid T'_p) = \left(\begin{array}{llll|llll} 1 & & & & 1 & & & \\ \cdot & & & & \cdot & & & \\ \cdot & & & & \cdot & & & \\ \cdot & 2 & & & \cdot & 2 & & \\ \vdots & \vdots & \cdot & & \vdots & \vdots & \cdot & \\ & & & \cdot & & & & \cdot \\ 1 & 2 & & \cdot & 1 & 2 & & \cdot \\ 1 & 2 & \dots & \mu_1 & 1 & 2 & \dots & \mu_1 \end{array}\right)$$

(only the subscripts are written).

In the expansion of this bideterminant there appears the term $\prod_{1\leq i\leq\mu_1}(s_i \mid t_i)^{\tilde{\mu}_i}$, where $\tilde{\mu}_i$ is the height of the ith column of (μ). Its coefficient is 1. This term occurs in $C_p(V \mid V')$ or in $C_p(T_i \mid T'_i)$ only if their normalized consist of two identical tableaux. The determination of a_p can so be reduced to the picking up of such bideterminants.

Example. Let us apply the straightening formula to the preceding example. We have to straighten the bideterminant

$$\left(\begin{array}{lll|lll} 1 & 4 & & 4 & 5 & \\ 2 & 3 & 5 & 1 & 2 & 3 \end{array}\right).$$

The bideterminants which appear in the decomposition must have shapes $(3,2)$, $(4,1)$ or (5). Those of shape $(3,2)$ can only be bideterminants of the form

$$\left(T_i \,\middle|\, \begin{array}{lll} 4 & 5 & \\ 1 & 2 & 3 \end{array}\right), \quad \text{where } T_i \geq \begin{array}{lll} 2 & 4 & \\ 1 & 3 & 5 \end{array},$$

which is the standardized of

$$\begin{array}{lll} 1 & 4 & \\ 2 & 3 & 5 \end{array}.$$

Five tableaux are to be found:

$$\begin{array}{lll} 2 & 4 & \\ 1 & 3 & 5 \end{array}, \quad \begin{array}{lll} 2 & 5 & \\ 1 & 3 & 4 \end{array}, \quad \begin{array}{lll} 3 & 4 & \\ 1 & 2 & 5 \end{array}, \quad \begin{array}{lll} 3 & 5 & \\ 1 & 2 & 4 \end{array} \quad \text{and} \quad \begin{array}{lll} 4 & 5 & \\ 1 & 2 & 3 \end{array}.$$

From Theorem 5 we have

$$\left(\begin{array}{lll|lll} 1 & 4 & & 4 & 5 & \\ 2 & 3 & 5 & 1 & 2 & 3 \end{array}\right) = \left(\begin{array}{lll|lll} 2 & 4 & & 4 & 5 & \\ 1 & 3 & 5 & 1 & 2 & 3 \end{array}\right) + A.$$

For the next coefficient, apply the Capelli operator of

$$\left[\begin{array}{ccc} 2 & 5 & \\ 1 & 3 & 4 \end{array}, \begin{array}{ccc} 4 & 5 & \\ 1 & 2 & 3 \end{array}\right]:$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & & 1 & 2 & \\ 1 & 2 & 2 & 1 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 3 & & 1 & 2 & \\ 1 & 2 & 2 & 1 & 2 & 3 \end{array}\right) + a\left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 2 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right);$$

the first two bideterminants are zero so that $a=0$. Do the same for

$$\left[\begin{array}{ccc} 3 & 4 & \\ 1 & 2 & 5 \end{array}, \begin{array}{ccc} 4 & 5 & \\ 1 & 2 & 3 \end{array}\right]:$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 2 & \\ 2 & 1 & 3 & 1 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|ccc} 2 & 2 & & 1 & 2 & \\ 1 & 1 & 3 & 1 & 2 & 3 \end{array}\right) + a\left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 2 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right),$$

and, after reduction:

$$-\left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 2 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right) = a\left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 2 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right) \quad \text{and} \quad a=-1.$$

we now have

$$\left(\begin{array}{ccc|ccc} 1 & 4 & & 4 & 5 & \\ 2 & 3 & 5 & 1 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|ccc} 2 & 4 & & 4 & 5 & \\ 1 & 3 & 5 & 1 & 2 & 3 \end{array}\right) - \left(\begin{array}{ccc|ccc} 3 & 4 & & 4 & 5 & \\ 1 & 2 & 5 & 1 & 2 & 3 \end{array}\right) + A.$$

The coefficient of

$$\left(\begin{array}{ccc|ccc} 3 & 5 & & 4 & 5 & \\ 1 & 2 & 4 & 1 & 2 & 3 \end{array}\right)$$

is 0, as it could be shown by an analogous argument. For the last bideterminant we obtain

$$\left(\begin{array}{ccc|ccc} 1 & 1 & & 1 & 2 & \\ 2 & 3 & 2 & 1 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|ccc} 2 & 1 & & 1 & 2 & \\ 1 & 3 & 2 & 1 & 2 & 3 \end{array}\right) - \left(\begin{array}{ccc|ccc} 3 & 1 & & 1 & 2 & \\ 1 & 2 & 2 & 1 & 2 & 3 \end{array}\right) + a\left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 2 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right),$$

after reduction:

$$0 = \left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 2 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right) + a\left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 2 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right) \quad \text{and} \quad a=-1.$$

Hence

$$\left(\begin{array}{ccc|ccc} 1 & 4 & & 4 & 5 & \\ 2 & 3 & 5 & 1 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|ccc} 2 & 4 & & 4 & 5 & \\ 1 & 3 & 5 & 1 & 2 & 3 \end{array}\right) - \left(\begin{array}{ccc|ccc} 3 & 4 & & 4 & 5 & \\ 1 & 2 & 5 & 1 & 2 & 3 \end{array}\right) - \left(\begin{array}{ccc|ccc} 4 & 5 & & 4 & 5 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right) + A, \qquad (*)$$

where A is a linear combination of strictly longer bideterminants.

Four standard tableaux of shape (4, 1) exist:

$$\begin{array}{cccc} 2 & & & \\ 1 & 3 & 4 & 5 \end{array}, \quad \begin{array}{cccc} 3 & & & \\ 1 & 2 & 4 & 5 \end{array}, \quad \begin{array}{cccc} 4 & & & \\ 1 & 2 & 3 & 5 \end{array} \quad \text{and} \quad \begin{array}{cccc} 5 & & & \\ 1 & 2 & 3 & 4 \end{array}.$$

We are not concerned by bideterminants whose right tableau is one of the first three: their Capelli operators cancel every term in $(*)$. To find our the coefficient of

$$\left(\begin{array}{cccc|cccc} 2 & & & & 4 & & & \\ 1 & 3 & 4 & 5 & 1 & 2 & 3 & 5 \end{array}\right),$$

which is the smallest possible in A, apply to $(*)$ its Capelli operator:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & & 1 & 4 & \\ 1 & 2 & 4 & 1 & 2 & 3 \end{array}\right)=\left(\begin{array}{ccc|ccc} 1 & 3 & & 1 & 4 & \\ 1 & 2 & 4 & 1 & 2 & 3 \end{array}\right)-\left(\begin{array}{ccc|ccc} 2 & 3 & & 1 & 4 & \\ 1 & 1 & 4 & 1 & 2 & 3 \end{array}\right)$$
$$+\left(\begin{array}{ccc|ccc} 3 & 4 & & 1 & 4 & \\ 1 & 1 & 2 & 1 & 2 & 3 \end{array}\right)+a\left(\begin{array}{cccc|cccc} 1 & & & & 1 & & & \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{array}\right).$$

The term $(s_1 \mid t_1)^2(s_2 \mid t_2)(s_3 \mid t_3)(s_4 \mid t_4)$ only appears in

$$\left(\begin{array}{cccc|cccc} 1 & & & & 1 & & & \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{array}\right)$$

so that $a=0$. In the same manner the coefficient of

$$\left(\begin{array}{cccc|cccc} 3 & & & & 4 & & & \\ 1 & 2 & 4 & 5 & 1 & 2 & 3 & 5 \end{array}\right)$$

is 0. For

$$\left(\begin{array}{cccc|cccc} 4 & & & & 4 & & & \\ 1 & 2 & 3 & 5 & 1 & 2 & 3 & 5 \end{array}\right),$$

we obtain, after reduction:

$$0=-\left(\begin{array}{ccc|ccc} 1 & 2 & & 1 & 4 & \\ 1 & 3 & 4 & 1 & 2 & 3 \end{array}\right)+\left(\begin{array}{ccc|ccc} 1 & 3 & & 1 & 4 & \\ 1 & 2 & 4 & 1 & 2 & 3 \end{array}\right)-\left(\begin{array}{ccc|ccc} 1 & 4 & & 1 & 4 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right)$$
$$+a\left(\begin{array}{cccc|cccc} 1 & & & & 1 & & & \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{array}\right).$$

The term $(s_1 \mid t_1)^2(s_2 \mid t_2)(s_3 \mid t_3)(s_4 \mid t_4)$ appears in the last two bideterminants; therefore $a=1$.

$$\left(\begin{array}{ccc|ccc} 1 & 4 & & 4 & 5 & \\ 2 & 3 & 5 & 1 & 2 & 3 \end{array}\right)=\left(\begin{array}{ccc|ccc} 2 & 4 & & 4 & 5 & \\ 1 & 3 & 5 & 1 & 2 & 3 \end{array}\right)-\left(\begin{array}{ccc|ccc} 3 & 4 & & 4 & 5 & \\ 1 & 2 & 5 & 1 & 2 & 3 \end{array}\right)$$
$$-\left(\begin{array}{ccc|ccc} 4 & 5 & & 4 & 5 & \\ 1 & 2 & 3 & 1 & 2 & 3 \end{array}\right)+\left(\begin{array}{cccc|cccc} 4 & & & & 4 & & & \\ 1 & 2 & 3 & 5 & 1 & 2 & 3 & 5 \end{array}\right)+A. \quad (**)$$

The same calculation with

$$\left(\begin{array}{cccc|cccc} 5 & & & & 4 & & & \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 5 \end{array}\right)$$

leads to

$$\begin{aligned}\left(\begin{array}{l|l}1\ 4 & 1\ 4\\ 2\ 3\ 1 & 1\ 2\ 3\end{array}\right) &= \left(\begin{array}{l|l}2\ 4 & 1\ 4\\ 1\ 3\ 1 & 1\ 2\ 3\end{array}\right) - \left(\begin{array}{l|l}3\ 4 & 1\ 4\\ 1\ 2\ 1 & 1\ 2\ 3\end{array}\right)\\ &\quad - \left(\begin{array}{l|l}4\ 1 & 1\ 4\\ 1\ 2\ 3 & 1\ 2\ 3\end{array}\right) + \left(\begin{array}{l|l}4 & 1\\ 1\ 2\ 3\ 1 & 1\ 2\ 3\ 4\end{array}\right)\\ &\quad + a\left(\begin{array}{l|l}1 & 1\\ 1\ 2\ 3\ 4 & 1\ 2\ 3\ 4\end{array}\right),\end{aligned}$$

which reduces to

$$\left(\begin{array}{l|l}1\ 4 & 1\ 4\\ 1\ 2\ 3 & 1\ 2\ 3\end{array}\right) = \left(\begin{array}{l|l}1\ 4 & 1\ 4\\ 1\ 2\ 3 & 1\ 2\ 3\end{array}\right) + a\left(\begin{array}{l|l}1 & 1\\ 1\ 2\ 3\ 4 & 1\ 2\ 3\ 4\end{array}\right);$$

consequently $a=0$.

For the bideterminants whose right tableau is

$$\begin{array}{l}5\\ 1\ 2\ 3\ 4\end{array},$$

the same operations can be performed or, more simply permute the 4 and the 5 in the right side of every bideterminant of $(**)$:

$$\begin{aligned}\left(\begin{array}{l|l}1\ 4 & 5\ 4\\ 2\ 3\ 5 & 1\ 2\ 3\end{array}\right) &= \left(\begin{array}{l|l}2\ 4 & 5\ 4\\ 1\ 3\ 5 & 1\ 2\ 3\end{array}\right) - \left(\begin{array}{l|l}3\ 4 & 5\ 4\\ 1\ 2\ 5 & 1\ 2\ 3\end{array}\right)\\ &\quad - \left(\begin{array}{l|l}4\ 5 & 5\ 4\\ 1\ 2\ 3 & 1\ 2\ 3\end{array}\right) + \left(\begin{array}{l|l}4 & 5\\ 1\ 2\ 3\ 5 & 1\ 2\ 3\ 4\end{array}\right) + A'.\end{aligned}$$

After normalization we have

$$\begin{aligned}\left(\begin{array}{l|l}1\ 4 & 4\ 5\\ 2\ 3\ 5 & 1\ 2\ 3\end{array}\right) &= \left(\begin{array}{l|l}2\ 4 & 4\ 5\\ 1\ 3\ 5 & 1\ 2\ 3\end{array}\right) - \left(\begin{array}{l|l}3\ 4 & 4\ 5\\ 1\ 2\ 5 & 1\ 2\ 3\end{array}\right)\\ &\quad - \left(\begin{array}{l|l}4\ 5 & 4\ 5\\ 1\ 2\ 3 & 1\ 2\ 3\end{array}\right) - \left(\begin{array}{l|l}4 & 5\\ 1\ 2\ 3\ 5 & 1\ 2\ 3\ 4\end{array}\right) - A';\end{aligned}$$

in A' no bideterminant whose right side is $\begin{array}{l}5\\ 1\ 2\ 3\ 4\end{array}$ occurs.

We have now:

$$\begin{aligned}\left(\begin{array}{l|l}1\ 4 & 4\ 5\\ 2\ 3\ 5 & 1\ 2\ 3\end{array}\right) &= \left(\begin{array}{l|l}2\ 4 & 4\ 5\\ 1\ 3\ 5 & 1\ 2\ 3\end{array}\right) - \left(\begin{array}{l|l}3\ 4 & 4\ 5\\ 1\ 2\ 5 & 1\ 2\ 3\end{array}\right)\\ &\quad - \left(\begin{array}{l|l}4\ 5 & 4\ 5\\ 1\ 2\ 3 & 1\ 2\ 3\end{array}\right) + \left(\begin{array}{l|l}4 & 4\\ 1\ 2\ 3\ 5 & 1\ 2\ 3\ 5\end{array}\right)\\ &\quad - \left(\begin{array}{l|l}4 & 5\\ 1\ 2\ 3\ 5 & 1\ 2\ 3\ 4\end{array}\right)\\ &\quad + a(1\ 2\ 3\ 4\ 5\,|\,1\ 2\ 3\ 4\ 5). \qquad (***)\end{aligned}$$

Regarding the coefficient of (1 2 3 4 5 | 1 2 3 4 5), we search for the term

$$(x_1 \mid u_1)(x_2 \mid u_2)(x_3 \mid u_3)(x_4 \mid u_4)(x_5 \mid u_5)$$

which appears in any bideterminant composed of two identical tableaux. The sum of the coefficients of the such bideterminants in $(***)$ is zero; thus $a=0$.

This concludes the determination of the straightening coefficients.

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